ON BONNET' S THEOREM

(O TEOREME BONNE)

PMM Vol.22, No.6, 1958, pp.721-729

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(Received 25 July 1957)

The generalization of the Bonnet Theorem presented in this paper proves the possibility of a motion of a particle along a curve, under the action of an equivalent [resultant] force (resultant motion), if motions under the action of component forces (partial motions) are also possible. With the help of the above generalization, certain theorems of Bonnet [1] Lagrange [2] and Tallkvist [3] on the motion along hyperbolic arcs in the problem of two fixed attracting centers are corrected and made more precise.

1. The generalization of Bonnet's Theorem. In 1844, Bonnet ([1], p. 13) proved the following theorem: If several masses m, m', m'', ..., acted on by forces F, F', F'', ..., respectively, with initial unequal codirectional velocities, v_0, v_0', v_0'', ..., with the same initial position A are tracing the same curve ABC, then a certain mass M acted on by forces F, F', F'', ..., with initial velocity V colinear with velocities v_0, v_0', v_0'', ..., at the initial position A will trace the same curve ABC, if the forces F, F', F'', ..., are independent of time and the initial kinetic energy MV_0^2 of the mass M equals the sum $mv_0^2 + m'v'_0{}^2 + m''v_0''^2 +$, ..., of the initial kinetic energies of the masses m, m', m''.

A generalization of the above theorem is as follows:

Theorem. Let each of the masses m_1, m_2, ..., m_n acted on by forces $F_i (i \;\; 1, 2, \ldots, n)$, which depend only on the position, trace a curve AB, and let v_{i0} be the velocity of the mass m_i at the initial point A. Then

1) A certain mass M, acted on by the force

$$\mathbf{F} = a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n$$

(where a_i is a constant), having velocity V_0 with the same direction as v_{i0} at the point A, will trace the same curve AB, or part of it, if, and only if

$$MV_0^2 = a_1 m_1 v_{10}^2 + \cdots + a_n m_n v_{n0}^2 > 0 \tag{1.1}$$

2) Conversely, if curve AB traced by mass M with velocity V acted on by the force

$$\mathbf{F} = a_1 \mathbf{F}_1 + \ldots + a_n \mathbf{F}_n$$

then along curve AB

$$MV^2 = a_1 m_1 v_1^2 + \cdots + a_n m_n v_n^2 \tag{1.2}$$

In this way forces $\mathbf{F}_i$ and the corresponding motions could be regarded as base forces and base motions by analogy with vector spaces. The original Bonnet theorem is a special case of this generalization, and is obtained by putting $a_i = +1$.

The equations of motion are linear with respect to masses and forces; therefore, the proof of the first part of the generalized theorem is almost identical with the proof of Bonnet.

Let us call the motion caused by one of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, ..., $\mathbf{F}_n$, acting alone, a partial motion, and the motion caused by force $\mathbf{F}$, the resultant motion.

If in the resultant motion mass M does not trace curve AB, then we shall apply a force $\mathbf{N}$, normal to the curve, such that the action of $\mathbf{F} + \mathbf{N}$ will cause M to trace curve AB. Then

$$M d\mathbf{V}/dt = a_1 \mathbf{F}_1 + \ldots + a_n \mathbf{F}_n + \mathbf{N} \tag{1.3}$$

where $\mathbf{V}$ is the velocity vector of mass M at any point C on the curve AB. Multiplying scalarly equation (1.3) by the vector of elementary displacement along the curve, $\mathbf{d}_s$, we obtain

$$dMV^2 = 2(a_1 \mathbf{F}_1 \cdot \mathbf{d}_s + \ldots + a_n \mathbf{F}_n \cdot \mathbf{d}_s) \tag{1.4}$$

If in the partial motion v_i is the velocity of the mass m_i ($i = 1, \ldots, n$) at a point C, then similarly to (1.4) for the same displacement $\mathbf{d}_s$ we have

$$dm_i v_i^2 = 2\mathbf{F}_i \cdot \mathbf{d}_s \qquad (i = 1, \ldots, n) \tag{1.5}$$

The intervals, corresponding to the fixed displacement $\mathbf{d}_s$, are different in the partial and resultant motions because of the different velocities v_i and V. The positiondependent forces $\mathbf{F}_i$, however, are the same in (1.4) and (1.5), and the corresponding scalar products (1.4) and (1.5) are equal at the same point C.

Multiplying equation (1.5) by a_i, adding together left-hand members and right-hand members, and substituting the resulting right-hand member in (1.4), we obtain

$$dMV^2 = a_1 d(m_1 v_1^2) + \ldots + a_n d(m_n v_n^2)$$

Integrating the above equation, by virtue of condition (1.1) we obtain the expression for MV^2 on curve AB

$$MV^2 = a_1 m_1 v_1^2 + \ldots + a_n m_n v_n^2 \qquad (1.6)$$

valid at any point of curve AB, and positive on curve AB at least in the neighborhood of point A (by virtue of the inequality in (1.1)).

We will now prove that along curve AB, force $\mathbf{N} \equiv 0$. In the plane π, normal to the curve, for each partial motion we have

$$\frac{m_i v_i^2}{\rho} \mathbf{n} = (\mathbf{F}_i)_\pi \qquad (= 1, \ldots, n) \qquad (1.7)$$

where $\mathbf{n}$ is the unit vector of the principal normal, ρ is the radius of curvature, and the subscript π indicates projection in the π-plane. Projecting (1.3) on π-plane, we obtain

$$\frac{MV^2}{\rho} \mathbf{n} = [a_1 (\mathbf{F}_1)_\pi + \ldots + a_n (\mathbf{F}_n)_\pi] + \mathbf{N} \qquad (1.8)$$

Again utilizing the fact that in the partial and resultant motions we have the same force $\mathbf{F}_i$, we will substitute (1.7) in (1.8), obtaining $\mathbf{N} = 0$ by virtue of (1.6); point C is arbitrary, hence $\mathbf{N}$ must identically equal zero, which was to be proved.

We will now prove the second part of the theorem. Projecting the equations of partial and resultant motions on the π-plane, we obtain (1.7) and (1.8), by virtue also of the conditions of the second part of the theorem $\mathbf{N} \equiv 0$. Once more using the equality of $\mathbf{F}_i$ in (1.7) and (1.8), and substituting from (1.7) $\mathbf{F}_i$ multiplied by a_i into (1.8), and then multiplying (1.8) by $(\rho\mathbf{n})$, we obtain an expression which proves the second part of the theorem.

The first part of the theorem does not tell whether the resultant motion along curve AB originating at A is possible when condition (1.1) is not satisfied. The second part of the theorem answers in the negative, for if such a motion were possible then condition (1.1) would be satisfied at every point of the curve.

The time independence of the forces $\mathbf{F}_i$, assumed by Bonnet is insufficient for the proof of the Bonnet and of the generalized theorem; it is necessary for forces $\mathbf{F}_i$ to be dependent on the position C only. Indeed, if the forces $\mathbf{F}_i$ depend on some parameters which at C are different for partial and resultant motions, for example velocity, then the theorem cannot be proved. On the other hand, forces $\mathbf{F}_i$ need not be purely positiondependent. For example, forces depending only on the trajectory curvature or on the direction of the velocity are in general not positiondependent, but are so when the trajectory is fixed. Moreover, forces $\mathbf{F}_i$ could be given only along curve AB but unspecified elsewhere.

If the forces depend not only on the position C, but also on the mass, that is if $\mathbf{F}_i = k_i(m)\mathbf{f}_i(C)$, then the generalized theorem remains valid if in the parts (1) and (2) of the theorem the mass m_i is replaced by $m_i k_i(M)/k_i(m_i)$. Masses completely cancel out in conditions (1.1) and (1.2) when $m_1 = m_2 = \ldots m_n$. This occurs when the forces are mass-forces.

The generalized theorem can also be proved for a constrained motion of a particle, when curve AB lies on a smooth surface. Indeed, the reaction forces $\mathbf{N}_0$ and $\mathbf{N}_{i0}$ in the partial and the resultant motions are respectively normal to the π_0-plane, which is tangent to the surface and to trajectory AB. The force $\mathbf{N}$ which constrains the particle to trace curve AB may lie in the π_0-plane. Now, instead of (1.3) we have

$$M d\mathbf{V}/dt = a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n + \mathbf{N} + \mathbf{N}_0 \tag{1.9}$$

The relations (1.4) and (1.6) obviously remain valid, and (1.7) will change into

$$\frac{m_i v_i^2}{\rho}\mathbf{n} = (F_i)_\pi + \mathbf{N}_{i0} \tag{1.10}$$

Projecting (1.9) on the π-plane, normal to the curve, we obtain

$$M\frac{V^2}{\rho}\mathbf{n} = a_1(\mathbf{F}_1)_\pi + \ldots + a_n(\mathbf{F}_n)_\pi + \mathbf{N} + \mathbf{N}_0 \tag{1.11}$$

Substituting in (1.11) the expressions for $(\mathbf{F}_i)_\pi$ from (1.10), we have

$$\left[\frac{MV^2}{\rho} - \frac{1}{\rho}(a_1 m_1 v_1^2 + \ldots + a_n m_n v_n^2)\right]\mathbf{n} = \mathbf{N} + \mathbf{N}_0 - (a_1\mathbf{N}_{10} + \ldots + a_n\mathbf{N}_{n0})$$

or, by virtue of (1.6)

$$0 = \mathbf{N} + \mathbf{N}_0 - (a_1\mathbf{N}_{10} + \ldots + a_n\mathbf{N}_{n0})$$

If we choose $\mathbf{N}$ orthogonal to $\mathbf{N}_0$ and $\mathbf{N}_{i0}$ then we have $\mathbf{N} = 0$, $\mathbf{N}_0 = a_1\mathbf{N}_{10} + \ldots + a_n\mathbf{N}_{n0}$; that is, we have not only proved the first part of the generalized theorem for a constrained motion, but have also obtained a simple expression for the normal reaction in the resultant motion in terms of normal reactions of the partial motions.

The proof of the second part of the theorem follows, if instead of equations (1.7) and (1.8) we use equations (1.10) and (1.11).

Note: It can easily be shown that the generalized theorem can be formulated and proved in the following symmetric form:

Let $(n-1)$ particles among a set of n particles trace the same curve AB or part of it, starting from point A with initial codirectional velocities v_{i0} and under the action of the positiondependent forces $\mathbf{F}_i (i=1, \ldots, n)$ respectively. Then

(1) The remaining mass m_k will trace the same curve, or part of it, if there exist real numbers $a_i (i = 1, \ldots, n)$ such that

$$a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n = 0, \qquad a_1 m_1 v_{10}^2 + \ldots + a_n m_n v_{n0}^2 = 0 \tag{1.12}$$

and the value of $m_k v_k^2$ derived from (1.2) satisfies $m_k v_k^2 > 0$.

(2) If curve AB is traced by the remaining mass m_k under the condition

$$a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n = 0$$

where a_i are real numbers, then at any point of the curve

$$a_1 m_1 v_1^2 + \ldots + a_n m_n v_n^2 = 0 \tag{1.13}$$

2. Example of an application of the generalized theorem. Suppose we wish to know whether or not a motion of a mass M on a curve AB under the action of the sum of the given forces $\Phi_1, \ldots, \Phi_n$ is possible.

If for every force Φ_i we can find a mass m_i and a constant a_i such that m_i would trace curve AB under action of force $\mathbf{F}_i = \Phi_i/a_i$, then the generalized theorem fully answers the question, for the motion of the mass m_i under the action of force $\mathbf{F}_i$ can be regarded as partial motion, and the motion of mass M under the action of the equivalent force

$$a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n = \Phi_1 + \ldots + \Phi_n$$

as the resultant motion. It appears that a force required for the partial motion can differ from a given force by a scalar factor. By introducing forces $\mathbf{F}_i$ differing from given forces Φ_i, we can analyse motions to which the original Bonnet theorem is not applicable. We will illustrate this by an example.

In the problem of motion of a particle gravitationally attracted by two fixed masses α and β, Lagrange [1] - using the elliptic coordinates

$$s = r + q \quad \text{and} \quad u = r - q\ ,$$

where r and q are distances of the particle from α and β respectively - reduced the problem to quadratures. He showed that the particular solutions of the problem are

$$s = s_0 \quad \text{and} \quad u = u_0$$

where s_0 and u_0 are multiple roots of the polynomials

$$S = S(s) \quad \text{and} \quad U = U(u),$$

which are under the radical sign in the denominators of the integrals. The solution $s = s_0$ is an ellipse with α and β as the foci, and the solution $u = u_0$ is one branch of a hyperbola with the same foci. Moreover, Lagrange states ([2], p. 129): "In this way, the particular solutions discussed above give ellipses or hyperbolas traced around the force centers α/r^2 and β/q^2, taken as the foci. Since the polynomials S and U contain three arbitrary constants A, B and C depending on the initial direction and initial velocity of the particle, it is clear that we can always choose these parameters in such a way that the particle will trace the prescribed ellipse or hyperbola with α and β as foci." Using the generalized theorem, we will prove that the above statement, valid for an ellipse with the foci α and β is not valid for any hyperbola with the foci $\alpha \neq \beta$, and we will show all those branches of hyperbola given by the solution $u = u_0$, which can be traced about the foci $\alpha \neq \beta$.

Legendre ([8], p. 426) also proved, independently of Lagrange, that an arbitrary ellipse with foci α and β is a solution of the problem.

In paper [1] already mentioned, Bonnet claimed (before formulating his theorem) that the above statements by Lagrange and Legendre follow from his theorem. Bonnet's claim must be corrected, because Lagrange's statement with respect to the hyperbolic solution $u = u_0$ does not follow directly from Bonnet's theorem. Indeed, a fixed branch of hyperbola with foci α and β does not satisfy the conditions of Bonnet's theorem; as its concavity is turned towards one of the attracting centers, it cannot be traced under the action of the second attractive center alone.

This was noticed in 1866 by Sylvester [4] who mentioned that Bonnet's theorem could be made applicable to hyperbolas by the introduction of negative kinetic energies and imaginary motions.

However, the generalization of Bonnet's theorem proved in Section 1 can be applied to hyperbolas without the use of imaginaries. By the introduction of repulsive forces for the partial motion, the problem of existence regions and other properties of purely hyperbolic motion can be fully solved by this generalization. Since the centers are always in the plane of the hyperbola, it is sufficient to resolve this question in one plane only.

Using polar coordinates r, θ, we will consider the motion of a particle under the action of a central repulsive force, inversely proportional to the square of the distance. The integrals of kinetic energy and area are respectively

$$\frac{V^2}{2} = -\frac{\mu}{r} + h \quad (\mu < 0), \qquad r^2 \frac{d\theta}{dt} = \text{const}$$

which are similar to the corresponding integrals in the case of an attracting mass $\mu > 0$, the difference being the sign preceding μ.

From these two integrals it is easy to derive the solution

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

which is similar to that in the case of an attracting mass $\mu > 0$, where p has the same sign as μ.

Since $\mu < 0$, the positive values of r, corresponding to real trajectories, exist only when $e > 0$, which means that we can have only hyperbolic motion (when $e < 1$, by virtue of $p < 0$ we have $r < 0$). Besides, the repulsing center is not the nearer focus but the distant one with respect to the branch traced. This proves quite convenient for the solution of the problem by the generalized theorem.

With the proper choice of units of mass and time, we can have the distance between the centers equal unity, the attraction constant equal unity, and $\alpha + \beta = 1$, i.e. $\alpha = 1 - \beta$. Let $\beta < \alpha$, that is $\beta < 0.5$ (In Fig. 1, $\beta = 0.1$).

Since a branch of the hyperbola with the focus β can be traced not only under the action of one attractive force $\mathbf{F}_\beta$, $|\mathbf{F}_\beta| = \beta/r_\beta^2$, attracting toward the focus β, but also under the action of one repulsive force $\mathbf{F}_\alpha$, $|\mathbf{F}_\alpha| = \alpha/r_\alpha^2$, repulsing from the focus α, and since the resultant force $\mathbf{F}$ in our problem of two attracting centers is

$$\mathbf{F} = \mathbf{F}_\beta + (-1)\,\mathbf{F}_\alpha$$

the first two motions can be regarded as partial motions, and the motion caused by the force $\mathbf{F}$ as a resultant motion. We will assume that the masses $m_1 = m_2 = M$, and the acting forces are mass forces. We can now apply the generalized theorem and determine where along the hyperbola the kinetic energy for the resultant motion is positive, that is, determine where the motion along a corresponding hyperbola is possible.

Applying the area and the kinetic energy integrals of the partial motions at the point C, which is the intersection of the given branch and the line $\alpha\beta$, and also at a point at infinity, we obtain

$$\frac{v_{\beta c}^2}{2} - \frac{\beta}{r_{\beta c}} = \frac{v_{\beta\infty}^2}{2}, \qquad r_{\beta c} v_{\beta c} = d v_{\beta\infty}$$

$$\frac{v_{\alpha c}^2}{2} + \frac{1-\beta}{r_{\alpha c}} = \frac{v_{\alpha\infty}^2}{2}, \qquad r_{\alpha c} v_{\alpha c} = d v_{\alpha\infty}$$

where d is the distance from the asymptote to the focus, and v_α and v_β are the corresponding velocities.

Introducing the angle γ between an asymptote and the line $\alpha\beta$, we have

$$r_{\beta c} = \tfrac{1}{2}(1 - \cos\gamma), \qquad r_{\alpha c} = \tfrac{1}{2}(1 + \cos\gamma), \qquad d = \tfrac{1}{2}\sin\gamma \tag{2.1}$$

Eliminating $v_{\alpha c}$ and $v_{\beta c}$ through area integrals, and taking into account (2.1) and then applying (1.2) for the resultant motion, we obtain

$$\frac{v_{\beta\infty}^2}{2} = \frac{\beta}{\cos\gamma}, \qquad \frac{v_{\alpha\infty}^2}{2} = \frac{1-\beta}{\cos\gamma}, \qquad \frac{V_\infty^2}{2} = \frac{2\beta - 1}{\cos\gamma} \tag{2.2}$$

Since $\beta < 1/2$ when $\gamma < \pi/2$, it follows for the velocity at infinity

$$1/2\ V_\infty^2 < 0,$$

which agrees with the second part of the theorem that the motion from infinity is impossible.

When $\gamma > \pi/2$, we have from (2.2) that

$$V_\infty^2 > 0;$$

hence for $\alpha \neq \beta$ the motion from infinity is possible, but only on a branch about the larger mass. In the latter case, when the branch approaches the line $r_\alpha = r_\beta$, that is when $\gamma \to \pi/2$, the quantity V_∞ approaches infinity. After finding kinetic energies for partial motions

$$T_\alpha = \frac{v_\alpha^2}{2} = -\frac{1-\beta}{r_\alpha} + \frac{1-\beta}{\cos\gamma}, \qquad T_\beta = \frac{v_\beta^2}{2} = \frac{\beta}{r_\beta} + \frac{\beta}{\cos\gamma}$$

we construct the function

$$T = \frac{1}{2}\left[v_\beta^2 + (-1)\, v_\alpha^2\right] = \frac{\beta}{r_\beta} + \frac{1-\beta}{r_\alpha} + \frac{2\beta - 1}{\cos\gamma} \tag{2.3}$$

The function T is obviously symmetric with respect to the point C and has a maximum at C. Using (2.1), we find the dependence of V_c on the orientation of a hyperbola:

$$T_c = \frac{V_c^2}{2} = \frac{2\beta(1 + \cos^2\gamma) - (1 - \cos\gamma)^2}{\cos\gamma \sin^2\gamma} \tag{2.4}$$

For $\gamma < \pi/2$

$$T_c \geqslant 0 \quad \text{when } \beta \geqslant \frac{1}{2}\,\frac{(1-\cos\gamma)^2}{1+\cos^2\gamma}$$

The above condition for β is satisfied for all $\gamma < \gamma^*(\beta)$, where the expression $\gamma^*(\beta)$ is obtained from the condition $T_c = 0$.

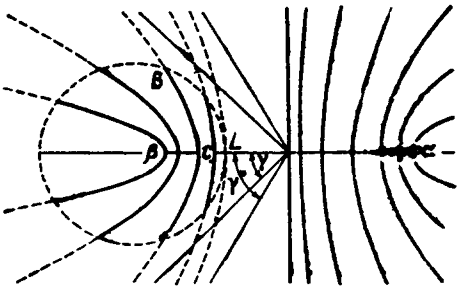

Fig. 1.

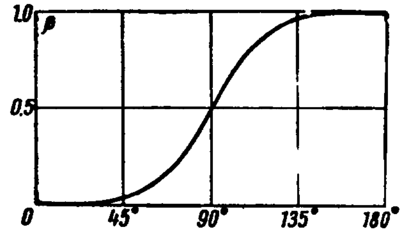

Fig. 2.

The graph of γ^* versus β is shown in Fig. 2, where γ^* is on the abscissas. It is seen that γ^* grows monotonically, approaching infinity asymptotically at the points $\beta = 0$ and $\beta = 1$. On the branch where $\gamma < \gamma^*$ near the point C, there exists a region where $T > 0$.

$$r_\alpha = a\cos\gamma, \quad r_\beta = b\cos\gamma, \quad a = \frac{\alpha + \sqrt{\alpha\beta}}{\alpha - \beta}, \quad b = \frac{\beta + \sqrt{\alpha\beta}}{\alpha - \beta} \qquad (2.5)$$

This region must be bounded by the locus of points $V = 0$. Substituting $T = 0$ and $r_\alpha = \cos\gamma + r_\beta$ in (2.3), which is obviously valid for the branch of hyperbola corresponding to the angle γ, we obtain

$$(2\beta - 1)(\cos\gamma + r_\beta)r_\beta + (1 - \beta)r_\beta\cos\gamma + \beta\cos\gamma(\cos\gamma + r_\beta) = 0$$

Solving the above quadratic equation, we find r_β and $r_\alpha = \cos\gamma + r_\beta$:

$$r_\beta = \frac{\beta \pm \sqrt{\beta(1-\beta)}}{1 - 2\beta}\cos\gamma, \qquad r_\alpha = \frac{1 - \beta \pm \sqrt{\beta(1-\beta)}}{1 - 2\beta}\cos\gamma$$

If we reject the minus sign, which gives $r_\beta < 0$ when $\beta < 1/2$ and $\gamma < \pi/2$, and replace $1 - \beta$ by α, then the above formulas reduce to the formulas (2.5).

In (2.5), neglecting $\cos\gamma$ and converting to Cartesian coordinates, we find that the curve defined by the parametric equations (2.5) is a circle of radius $\sqrt{\alpha\beta}/(\alpha - \beta)$, in which the distance between the center and the point β coincides with the line $\alpha\beta$ and equals $\beta/(\alpha - \beta)$.

Motions along the hyperbolic arcs are possible only inside this circle.

A body starting with zero velocity from the point A on the circumference will perform oscillatory motion on the hyperbolic arc ABC about its vertex C (Fig. 1).

The amplitude becomes maximum at $\gamma = 0$, decreasing and approaching zero as $\gamma \to \gamma^*$. We will prove that the zero amplitude corresponds to the libration point L, that is, to the point where the attracting masses α and β balance each other (Fig. 1).

Thus the libration point is found from the conditions:

$$r_\beta + r_\alpha = 1, \qquad \frac{\alpha}{r_\alpha^2} = \frac{\beta}{r_\beta^2} \quad \text{or} \quad \frac{r_\beta}{r_\alpha} = \sqrt{\frac{\beta}{\alpha}}.$$

The above condition restates the characteristic property of the circle (2.5); hence, when $r_\alpha + r_\beta = 1$, the circle passes through the libration point. It can easily be shown that inside the circle the attraction of mass β is stronger than the attraction of mass α.

Thus, a motion along hyperbolic arcs between the libration point and the line $r_\alpha = r_\beta$, that is, along the arcs where $\gamma^*(\beta) < \gamma < \pi/2$, would require negative kinetic energy and is therefore impossible. Motions along other hyperbolic arcs with foci α and β are possible everywhere in the half-plane $r_\beta \geqslant r_\alpha$, whereas in the half-plane $r_\beta < r_\alpha$ they are possible only inside the circle (2.5). This result conflicts with Lagrange's statement on the possibility of motion along any branch of a hyperbola.

There is only one special case, $\beta = \alpha = 1/2$, when motions are possible on any branch of a hyperbola with foci α and β. In this special case the velocity at infinity $V = 0$. When β approaches α, the circle (2.5) approaches the line $r_\beta = r_\alpha$, the region $T \geqslant 0$ becomes unbounded, and all zero-velocity points, except the libration point, recede to infinity. On account of the symmetry of the force field, the oscillatory motions along the line $r_\alpha = r_\beta$ can have arbitrary amplitudes, and velocities at infinity may assume any numerical value.

Remarks. 1. In this second paper ([1], p. 233) Bonnet gave a new proof of his theorem (formulated less generally than in 1 above) in which he again neglected its application to the hyperbolic solutions and also failed to notice that the theorem could be considerably generalized and made much more exact.

Witteker, who presented Bonnet's theorem in his book [5], also overlooked the possibility of greater generalization and exactness. He formulated Bonnet's theorem (Section 51) through purely positiondependent force fields, thus avoiding the inaccuracy contained in the original Bonnet formulation.

Nevertheless, in discussing the problem of two fixed centers in Section 53, Witteker applies Bonnet's theorem to confocal ellipses and hyperbolas, obviously not realizing that the theorem does not apply to hyperbolas.

Badalian [6,7], one of the later authors interested in the problem of two fixed attracting centers, makes the same mistake in applying Bonnet's theorem.

2. Badalian classifies all possible kinds of motion in the problem of two fixed centers, showing in particular two classes of motion along hyperbolas (for $h > 0$ and $h < 0$, where h is the constant kinetic energy), but he does not derive regions where motions of a given class can exist.

The possibility of oscillatory motions along hyperbolas is not a new discovery; it was noticed by Legendre ([8], p. 511), who briefly mentions that a condition for an oscillatory motion is that the velocity should depend on the position on the line a. Legendre made no detailed study of this dependence, did not derive the regions of existence, and failed to notice that a motion satisfying his condition is not necessarily oscillatory, but may also be a non-oscillatory motion along a hyperbola to infinity.

The two kinds of hyperbolic motions were first pointed out by Charlier [9], who classified all possible motions and showed the relation between the initial energies and the position. Charlier too does not analyse this relation, only mentioning that oscillatory motions along hyperbolas occur when $h < 0$, receding to infinity when $h > 0$.

3. Among the many papers on the problem of two fixed centers, there is only one study in which the derivation of existence regions for hyperbolic motion with foci α and β is attempted, namely that by Tallkvist [3], who discusses the problem of two centers for more than 500 pages, Using coordinates λ and μ deduced from the expressions $r_\alpha = \lambda + \mu$, $r_\beta = \lambda - \mu$, when $\mu_0 > 0$, for the oscillatory motion along a hyperbola Tallkvist obtains the relation between the initial energy $h < 0$ and the position in the form

$$\left(\frac{d\lambda}{dt}\right)_0^2 = \frac{\lambda_0^2 - c^2}{\mu_0}\left\{-\frac{m_1}{(\lambda_0+\mu_0)^2} + \frac{m_2}{(\lambda_0-\mu_0)^2}\right\}$$

where $2c$ is the distance between the particles $m_1 = \alpha$ and $m_2 = \beta$. Tallkvist makes the correct conclusion that such motions are possible only when

$$\frac{\lambda_0 - \mu_0}{\lambda_0 + \mu_0} < \sqrt{\frac{m_2}{m_1}} \tag{2.6}$$

and labels the case V_{kb}. It is, of course, clear that under the conditions (2.6) a hyperbola must pass between L and β (Fig. 1), which was to be expected.

But for the case labelled V_{ka}, that is, for hyperbolic motions with $h > 0$, $\mu_0 < 0$, Tallkvist obtains the erroneous (in sign) formula again leading to the condition (2.6), which is wrong for $h > 0$, $\mu_0 < 0$ (when $\mu_0 < 0$, then the left-hand member of (2.6) cannot be less than the right-hand member).

$$\left(\frac{d\lambda}{dt}\right)_0^2 = + \frac{\lambda_0^2 - c^2}{\mu_0}\left\{\frac{m_1}{(\lambda_0 + \mu_0)^2} - \frac{m_2}{(\lambda_0 - \mu_0)^2}\right\}$$

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Translated by T.L.